

Linear continuous surjections of C_p -spaces over compacta

Kazuhiro Kawamura and Arkady Leiderman

Abstract

Let X and Y be compact Hausdorff spaces and suppose that there exists a linear continuous surjection $T : C_p(X) \rightarrow C_p(Y)$, where $C_p(X)$ denotes the space of all real-valued continuous functions on X endowed with the pointwise convergence topology. We prove that $\dim X = 0$ implies $\dim Y = 0$. This generalizes a previous theorem [7, Theorem 3.4] for compact metrizable spaces. Also we point out that the function space $C_p(P)$ over the pseudo-arc P admits no densely defined linear continuous operator $C_p(P) \rightarrow C_p([0, 1])$ with a dense image.

1 Introduction and Results

For a Tychonoff space X , $C_p(X)$ denotes the space of all continuous real-valued functions on X endowed with the pointwise convergence topology. The relationship between the topology of X and linear topological properties of $C_p(X)$ is a subject of extensive research. A theorem of Pestov [13] plays the fundamental role in this study: if $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic for Tychonoff spaces X and Y , then we have the equality $\dim X = \dim Y$. The theorem was first proved by Pavlovskii for compact metrizable spaces [12]. A natural question arises whether the inequality

The authors are supported by JSPS KAKENHI Grant Number 26400080. The visit of the second author to University of Tsukuba in August, 2015 was supported by the grant above.

Keywords: C_p -theory, linear operators, dimension, hereditarily indecomposable continua
MSC. 54C35, 46E10, 54F45

$\dim Y \leq \dim X$ holds for Tychonoff spaces X and Y whenever there exists a linear continuous surjection $C_p(X) \rightarrow C_p(Y)$ [2, Problem 1046,1047]. This was answered negatively in [8],[7] even for compact metrizable spaces and the results were recently refined in [9]. The exception is the zero-dimensional case [7, Theorem 3.4]: if there exists a linear continuous surjection $C_p(X) \rightarrow C_p(Y)$ for compact metrizable spaces X and Y , then $\dim X = 0$ implies $\dim Y = 0$. The present paper extends the above theorem to all compact Hausdorff spaces.

Theorem 1.1 *Let X and Y be compact Hausdorff spaces and suppose that there exists a linear continuous surjection $T : C_p(X) \rightarrow C_p(Y)$. If $\dim X = 0$, then we have $\dim Y = 0$.*

Our proof is based on the spectral theorem of Shchepin [14] [3] which allows us to reduce our consideration to that on compact metrizable spaces (Proposition 2.2). Then a slight modification of [7, Theorem 3.4] supplies the desired result (Proposition 2.1).

The proof of Proposition 2.1 is applied to obtain more information on the existence of linear continuous surjections. It is known that for each finite- dimensional compact metrizable space X , there exists a linear continuous surjection $C_p([0, 1]) \rightarrow C_p(X)$ [8] and the map may be constructed to be open [9]. The assumption of finite- dimensionality cannot be dropped since there exists no linear continuous surjection $C_p([0, 1]) \rightarrow C_p([0, 1]^\omega)$ [8, Remark 4.6]. We see from the next proposition that $[0, 1]$ in these results cannot be replaced by the pseudo-arc P , the topologically unique hereditarily indecomposable continuum which is the limit of an inverse sequence of $[0, 1]$ (see a survey article [10]). Here a continuum means a compact connected metrizable space and a continuum X is said to be *hereditarily indecomposable* if each subcontinuum is not the union of two proper subcontinua. A compact metrizable space is called a *Bing compactum* if each connected component is either hereditarily indecomposable or is a singleton. A compact metrizable space Y is said to be *hereditarily locally connected* if each subcontinuum of Y is locally connected. Examples are $[0, 1]$ and more generally dendrites ([11]).

Proposition 1.2 *Let X be a Bing compactum, Y be a hereditarily locally connected compact metrizable space and let $T : C_p(X) \rightarrow C_p(Y)$ be a densely defined linear continuous operator with a dense image. Then we*

have $\dim Y = 0$. In particular there exists no linear continuous surjection $C_p(P) \rightarrow C_p([0, 1])$.

For a linear operator $T : E \rightarrow F$, $D(T)$ and $R(T)$ denote the domain of T and the image of T respectively. An operator $T : E \rightarrow F$ is said to be *densely defined* if $D(T)$ is dense in E . For a continuous map $\varphi : X \rightarrow Y$, the induced operator $\varphi^\# : C_p(Y) \rightarrow C_p(X)$ is defined by

$$\varphi^\#(f) = f \circ \varphi, \quad f \in C_p(Y).$$

2 Proofs

Proposition 2.1 *Let X and Y be compact metrizable spaces and let $T : E \rightarrow F$ be a linear continuous surjection defined on a dense subspace E of $C_p(X)$ onto a dense subspace F of $C_p(Y)$. If $\dim X = 0$, then we have $\dim Y = 0$.*

The compactness assumption cannot be dropped in the above theorem as is demonstrated by the following example.

Example 2.2 *There exists a densely defined surjective linear operator $T : C_p(C') \rightarrow C_p([0, 1])$, where C' is a G_δ subset of the Cantor set C and hence is a Polish space.*

Let $\varphi : C \rightarrow [0, 1]$ be a continuous 2-to-1 map of the Cantor set C onto $[0, 1]$ such that the set $\{y \in [0, 1] \mid |\varphi^{-1}(y)| = 2\}$ is at most countable. We then obtain a subset C' of C , by removing a countable subset from C , such that $\varphi' = \varphi|_{C'} : C' \rightarrow [0, 1]$ is a continuous bijection. The map induces a dense embedding $(\varphi')^\# : C_p([0, 1]) \rightarrow C_p(C')$ ([1, 0.4.8]) which naturally defines a densely defined operator $T : C_p(C') \rightarrow C_p([0, 1])$ with domain $R((\varphi')^\#) := (\varphi')^\#(C_p([0, 1]))$ onto $C_p([0, 1])$.

Proof of Proposition 2.1. Our proof is a modification of [7, Theorem 3.4] and is based on an analysis of the dual spaces. We first recall basics of the dual space notions [1, Chap.0]. For a Tychonoff space X , $L_p(X)$ denotes the dual space, that is, the space of all continuous linear functionals on

$C_p(X)$ endowed with the pointwise convergence topology. The space $L_p(X)$ is linearly homeomorphic to the space

$$\left\{ \sum_{i=1}^n \alpha_i x_i \mid \alpha_i \in \mathbb{R}, x_i \in X, n \in \mathbb{Z}_{\geq 0} \right\},$$

where the topology is described below. Each non-zero point x of $L_p(X)$ is uniquely written as

$$x = \sum_{i=1}^n \alpha_i x_i \quad (2.1)$$

where $\alpha_i \in \mathbb{R}^* := \mathbb{R} \setminus \{0\}$ and x_1, \dots, x_n are mutually distinct points of X . The number n above is called the length of x and is denoted by $\ell(x)$. Let $A_n(X) = \{x \in L_p(X) \mid \ell(x) = n\}$ and $B_n(X) = \{x \in L_p(X) \mid \ell(x) \leq n\}$. By the definition we have $A_n(X) = B_n(X) \setminus B_{n-1}(X)$ and also we have the equality

$$L_p(X) = \cup_{n=1}^{\infty} A_n(X). \quad (2.2)$$

It is known that each $B_n(X)$ is closed in $L_p(X)$ and each point $x = \sum_{i=1}^n \alpha_i x_i$ as in (2.1) has a neighborhood basis consisting of the sets

$$\sum_{i=1}^n O_i U_i$$

where O_i 's are open neighborhoods of α_i 's in \mathbb{R}^* and U_i 's are open neighborhoods of x_i 's respectively such that $U_i \cap U_j = \emptyset$ whenever $i \neq j$. It follows from this that the map

$$\sigma_n : (\mathbb{R}^*)^n \times (X^n \setminus \Delta_n) \rightarrow A_n(X), \quad ((\alpha_i), (x_i)) \mapsto \sum_{i=1}^n \alpha_i x_i, \quad (2.3)$$

where $\Delta_n = \{(x_1, \dots, x_n) \mid x_i = x_j \text{ for some } i \neq j\}$, is a homeomorphism.

Starting the proof of Proposition 2.1, let $\xi : E \rightarrow C_p(X)$ and $\eta : F \rightarrow C_p(Y)$ be the dense inclusions and consider the dual maps $\xi^* : L_p(X) \rightarrow E^*$ and $\eta^* : L_p(Y) \rightarrow F^*$. These maps ξ^* and η^* are continuous bijections because of the denseness of E and F . It follows from this that for each compact set K of $L_p(X)$, the restriction $\xi^*|_K : K \rightarrow \xi^*(K)$ is a homeomorphism. The same holds for compact sets of $L_p(Y)$. The subset Δ_n above is closed and thus G_δ in a metrizable compact space X^n . Then by (2.3), we see

that $A_n(X)$ is σ -compact and is represented as the union $A_n(X) = \cup_{i=1}^{\infty} A_{n,i}$, where each $A_{n,i}$ is homeomorphic to a compact subset of $(\mathbb{R}^*)^n \times X^n$. By the above remark, we have

$$\xi^*(A_n(X)) = \cup_{i=1}^{\infty} \xi^*(A_{n,i})$$

and each $\xi^*(A_{n,i})$ is homeomorphic to $A_{n,i}$.

The composition $T^* \circ \eta^* : L_p(Y) \rightarrow E^*$ embeds Y into E^* and hence Y is homeomorphic to the subspace

$$T^* \eta^*(Y) \subset \cup_{n,i=1}^{\infty} \xi^*(A_{n,i}).$$

Thus we have $Y = \cup_{k=1}^{\infty} Y_k$ where Y_k is a compact set such that the restriction

$$(\xi^*|_{A_{n(k),i(k)}})^{-1} \circ T^* \circ \eta^*|_{Y_k} : Y_k \rightarrow Y'_k$$

is a homeomorphism of Y_k onto a compact subset Y'_k of $A_{n(k),i(k)}$ for some $n(k)$ and $i(k)$. Below we show that $\dim Y_k = 0$ from which the desired conclusion follows by the countable sum theorem [5, Theorem 3.1.8].

Let $\tilde{Y}_k = \sigma_{n(k)}^{-1}(Y'_k)$ where $\sigma_{n(k)}$ is the homeomorphism of (2.3). Let $p : (\mathbb{R}^*)^{n(k)} \times X^{n(k)} \rightarrow X^{n(k)}$ be the projection and consider the restriction $p_k := p|_{\tilde{Y}_k}$. As a map defined on the compact set \tilde{Y}_k , the map p_k is a closed map onto a zero-dimensional subspace $p_k(\tilde{Y}_k)$ of $X^{n(k)}$. We show that p_k is at-most- $n(k)$ -to-1 map, that is,

$$|p_k^{-1}(x)| \leq n(k) \quad (2.4)$$

for each $x \in X^n$. Once (2.4) is verified, we obtain the equality $\dim \tilde{Y}_k = 0$ from the zero-dimensionality of $p_k(\tilde{Y}_k)$ by the dimension lowering theorem [5, Theorem 1.12.4], and therefore we conclude $\dim Y_k = 0$, as desired.

In what follows, $n(k)$ and p_k are simply denoted by n and p respectively. In order to verify (2.4), take a point $x = (x_1, \dots, x_n)$ of X^n and suppose on the contrary that the fiber $p^{-1}((x_1, \dots, x_n))$ contains $(n+1)$ points y^1, \dots, y^{n+1} of \tilde{Y}_k . For each $j = 1, \dots, n+1$, we may find $\lambda_{ij} \in \mathbb{R}^*$ such that

$$y^j = \sum_{i=1}^n \lambda_{ij} x_i. \quad (2.5)$$

By the definition of \tilde{Y}_k , the definitions of $\sigma_{n(k)}$ and the duality, the above (2.5) is rephrased as follows: for each $f \in E$ we have

$$(Tf)(y^j) = \sum_{i=1}^n \lambda_{ij} f(x_i), \quad j = 1, \dots, n+1. \quad (2.6)$$

Consider the matrix $\Lambda = (\lambda_{ij})_{1 \leq i, j \leq n}$ of size $n \times n$. The denseness of F with (2.6) implies that the linear map

$$\mathbf{v} \mapsto \Lambda \mathbf{v}, \quad \mathbb{R}^n \rightarrow \mathbb{R}^n$$

has the dense image. This implies that $\det \Lambda \neq 0$. In particular we may find $a_{ij} \in \mathbb{R}$ such that

$$(Tf)(y^{n+1}) = \sum_{i=1}^n a_{ij} Tf(y^j).$$

Then the image of the evaluation map

$$e : F \rightarrow \mathbb{R}^{n+1}; \quad e(g) = (g(y^j))_{1 \leq j \leq n+1}$$

is contained in an n -dimensional subspace of \mathbb{R}^{n+1} . However since F is dense in $C_p(Y)$, the set $e(F)$ must be dense in \mathbb{R}^{n+1} . This contradiction finishes the proof of (2.4) and thus finishes the proof of Proposition 2.1.

Here we recall some basics on inverse spectra from [3, Chap.1]. Let $\mathcal{S}_X = \{X_\alpha, p_{\alpha\beta}; A\}$ be an inverse system of topological spaces X_α , indexed by a directed set A with the limit space $X = \lim_{\leftarrow} \mathcal{S}_X$. The canonical projection of X to X_α is denoted by $p_\alpha : X \rightarrow X_\alpha$. An inverse system $\mathcal{S}_X = \{X_\alpha, p_{\alpha\beta}; A\}$ is called a *factorizing ω -spectrum* if

- (O1) each countable chain C of A has the supremum $\sup C \in A$,
- (O2) for each countable chain B of A with $\beta = \sup B$, the canonical map $\Delta_{\alpha \in B} p_{\alpha, \beta} : X_\beta \rightarrow \lim_{\leftarrow} \{X_\alpha, p_{\alpha_1 \alpha_2}; B\}$ is a topological embedding, and
- (F) each continuous function $f : X = \lim_{\leftarrow} \mathcal{S}_X \rightarrow \mathbb{R}$ admits an $\alpha \in A$ and $f_\alpha : X_\alpha \rightarrow \mathbb{R}$ such that $f = f_\alpha \circ p_\alpha$.

For a compact Hausdorff space X , there exists a factorizing ω -spectrum $\mathcal{S}_X = \{X_\alpha, p_{\alpha\beta}; A\}$ with $|A| \leq w(X)$ such that $X = \lim_{\leftarrow} \mathcal{S}_X$ and

- (C1) each X_α is a compact metrizable space,
- (C2) each limit projection p_α as well as each bonding map $p_{\alpha\beta}$ is surjective.
- (C3) the canonical map $\Delta_{\alpha \in B} p_{\alpha\beta}$ of (O2) is a surjective homeomorphism.

If $\dim X \leq n$, then we may choose above spectrum so that $\dim X_\alpha \leq n$ for each $\alpha \in A$ [3, Propositions 1.3.5, 1.3.2, and 1.3.10]. For a compact metrizable space X , the above spectrum is reduced to the trivial system $\{X, \text{id}_X\}$.

Proposition 2.3 *Let $X = \lim_{\leftarrow} \mathcal{S}_X$ and $Y = \lim_{\leftarrow} \mathcal{S}_Y$ be compact Hausdorff spaces which are the limits of factorizing ω -spectra $\mathcal{S}_X = \{X_\alpha, p_{\alpha_1\alpha_2}; A\}$ and $\mathcal{S}_Y = \{Y_\beta, p_{\beta_1\beta_2}; B\}$ satisfying the conditions (C1)-(C3) with the projections $p_\alpha : X \rightarrow X_\alpha$ and $q_\beta : Y \rightarrow Y_\beta$ respectively.*

Let $T : C_p(X) \rightarrow C_p(Y)$ be a linear continuous operator.

- (1) *For each α , there exist $\beta = \beta(\alpha)$ and a densely defined operator $T_{\alpha,\beta} : C_p(X_\alpha) \rightarrow C_p(Y_\beta)$ such that*

$$T \circ p_\alpha^\# = q_\beta^\# \circ T_{\alpha,\beta}.$$

For each $\beta_0 \in B$, we may choose the above $\beta(\alpha)$ so that $\beta(\alpha) \geq \beta_0$.

- (2) *If moreover T is surjective, then for each $\alpha_0 \in A$ and for each $\beta_0 \in B$, we may choose $T_{\alpha,\beta}$ so that $\alpha \geq \alpha_0, \beta \geq \beta_0$ and $T_{\alpha,\beta}$ has a dense image.*

The following diagram illustrates the operator $T_{\alpha,\beta}$.

$$\begin{array}{ccc} C_p(X) & \xrightarrow{T} & C_p(Y) \\ p_\alpha^\# \uparrow & & \uparrow q_\beta^\# \\ C_p(X_\alpha) & \xrightarrow{T_{\alpha,\beta}} & C_p(Y_\beta) \end{array}$$

Proof. Let $f_\alpha \in C_p(X_\alpha)$ and consider the composition $f_\alpha \circ p_\alpha$ whose image by T is factorized as

$$T(f_\alpha \circ p_\alpha) = g_\beta \circ q_\beta \tag{2.7}$$

for some β and $g_\beta \in C_p(Y_\beta)$. Observe that we may choose the above β as large as we wish. An important observation here is that, because q_β is surjective,

$$g_\beta \text{ is uniquely determined by } f_\alpha \text{ and } \beta. \tag{2.8}$$

Fix an arbitrary α and notice that $C_p(X_\alpha)$ is separable ([1, Theorem I.1.5]). Take a countable dense set $D = \{f_{\alpha,i}\}$ of $C_p(X_\alpha)$. For each i take a β_i and $g_i \in C_p(Y_{\beta_i})$ such that

$$T(f_{\alpha,i} \circ p_\alpha) = g_i \circ q_{\beta_i}.$$

Let $\beta = \sup_i \beta_i \in B$. By the ω -continuity of the spectrum (C3) we have

$$Y_\beta = \lim_{\leftarrow} \{Y_{\beta_i}, q_{\beta_i, \beta_j}\}$$

with the projection $q_{\beta_i, \beta} : Y_\beta \rightarrow Y_{\beta_i}$. For each i , the function $g_{\beta,i} = q_{\beta_i, \beta}^\#(g_i) \in C_p(Y_\beta)$ satisfies

$$T(f_{\alpha,i} \circ p_\alpha) = g_{\beta,i} \circ q_\beta.$$

Define $T_{\alpha, \beta} : D \rightarrow C_p(Y_\beta)$ by

$$T_{\alpha, \beta}(f_{\alpha,i}) = g_{\beta,i}$$

and let $D(T_{\alpha, \beta}) = \text{span} D$ which is a dense subspace of $C_p(X_\alpha)$. We make use of the uniqueness (2.8) to extend $T_{\alpha, \beta}$ to a linear operator $T_{\alpha, \beta} : D(T_{\alpha, \beta}) \rightarrow C_p(Y_\beta)$ as follows.

For $f = \sum_i \lambda_i f_{\alpha,i} \in D(T_{\alpha, \beta})$, all but finitely many λ_i 's being zero, let $T_{\alpha, \beta}(f) = \sum_i \lambda_i g_{\beta,i}$. Then $T_{\alpha, \beta}$ is well-defined and a linear map. Indeed, suppose that $\sum_i \lambda_i f_{\alpha,i} = \sum_i \mu_i f_{\alpha,i}$. Then we have the equality $\sum_i \lambda_i f_{\alpha,i} \circ p_\alpha = \sum_i \mu_i f_{\alpha,i} \circ p_\alpha$ and hence we have that $T(\sum_i \lambda_i f_{\alpha,i} \circ p_\alpha) = T(\sum_i \mu_i f_{\alpha,i} \circ p_\alpha)$. By linearity of T we see that

$$\begin{aligned} \sum_i \lambda_i g_{\beta,i} \circ q_\beta &= \sum_i \lambda_i T(f_{\alpha,i} \circ p_\alpha) = T(\sum_i \lambda_i f_{\alpha,i} \circ p_\alpha) \\ &= T(\sum_i \mu_i f_{\alpha,i} \circ p_\alpha) = \sum_i \mu_i T(f_{\alpha,i} \circ p_\alpha) \\ &= \sum_i \mu_i g_{\beta,i} \circ q_\beta, \end{aligned}$$

from which we see that $\sum_i \lambda_i g_{\beta,i} = \sum_i \mu_i g_{\beta,i}$ by the surjectivity of q_β . This proves that $T_{\alpha, \beta}$ is well-defined. The same argument is applied to prove that $T_{\alpha, \beta}$ is a linear map.

Finally we verify the continuity of $T_{\alpha, \beta}$. For a finite set F of a space Z , for an $\epsilon > 0$ and for a function $h \in C_p(Z)$, let

$$\langle h, F, \epsilon \rangle = \{u \in C(Z) \mid |u(p) - h(p)| < \epsilon \text{ for each } p \in F\}.$$

Fix an $\epsilon > 0$ and a finite subset F_β of Y_β . For each $y_\beta \in F_\beta$, take $y \in Y$ such that $q_\beta(y) = y_\beta$ by the surjectivity of q_β . Let F be the resulting finite subset of Y . By the continuity of T , we may find a finite subset E of X and $\delta > 0$ such that

$$T(< f_\alpha \circ p_\alpha, E, \delta >) \subset < T(f_\alpha \circ p_\alpha), F, \epsilon > .$$

Let $E_\alpha = p_\alpha(E)$. We verify the inclusion

$$T_{\alpha,\beta}(< f_\alpha, E_\alpha, \delta > \cap D(T_{\alpha,\beta})) \subset < T_{\alpha,\beta}(f_\alpha), F_\beta, \epsilon > .$$

Indeed for $f'_\alpha \in < f_\alpha, E_\alpha, \delta >$, we have $|f'_\alpha(p_\alpha(x)) - f_\alpha(p_\alpha(x))| < \delta$ for each $x \in E$ and hence $f'_\alpha \circ p_\alpha \in < f_\alpha \circ p_\alpha, E, \delta >$. For each $y \in F$ we have

$$|T(f'_\alpha \circ p_\alpha)(y) - T(f_\alpha \circ p_\alpha)(y)| < \epsilon,$$

which implies

$$|T_{\alpha,\beta}(f'_\alpha)(y_\beta) - T_{\alpha,\beta}(f_\alpha)(y_\beta)| < \epsilon$$

for each $y_\beta \in F_\beta$ by the equality $T_{\alpha,\beta}(f_\alpha) \circ q_\beta = T(f_\alpha \circ p_\alpha)$. Thus we see that $T_{\alpha,\beta}(f'_\alpha) \in < T_{\alpha,\beta}(f_\alpha), F_\beta, \epsilon >$. This proves the continuity of $T_{\alpha,\beta}$ and hence completes the proof of (1).

(2) Now assume that T is surjective. For an arbitrary α with $\alpha \geq \alpha_0$, take $\beta = \beta(\alpha) \geq \beta_0$ and $T_{\alpha,\beta} : C_p(X_\alpha) \rightarrow C_p(Y_\beta)$ as in (1). The operator $T_{\alpha,\beta}$ is defined on a dense subspace $D(T_{\alpha,\beta})$ of $C_p(X_\alpha)$ and satisfies

$$T \circ p_\alpha^\# = q_\beta^\# \circ T_{\alpha,\beta}. \quad (2.9)$$

Take a countable dense subset $E = \{g_i\}$ of $C_p(Y_\beta)$. Since T is surjective, for each i there exists $f_i \in C_p(X)$ such that $T(f_i) = g_i \circ q_\beta$. The function f_i factorizes as

$$f_i = f_{\alpha_i} \circ p_{\alpha_i}$$

for some $\alpha_i \in A$. Let $\alpha(1) = \sup_i \alpha_i$. The function $f_{\alpha(1),i} = f_{\alpha_i} \circ p_{\alpha_i \alpha(1)}$ satisfies $f_i = f_{\alpha(1),i} \circ p_{\alpha(1)}$ and hence

$$T(f_{\alpha(1),i} \circ p_{\alpha(1)}) = g_i \circ q_\beta. \quad (2.10)$$

Choose a countable dense subset D_1 of $C_p(X_{\alpha(1)})$ such that $D_1 \supset p_{\alpha,\alpha(1)}^\#(D) \cup \{f_{\alpha(1),i}\}$. Repeat the procedure of (1) with the use of D_1 to find $\beta(1) > \beta$

and a linear map $T_{\alpha(1),\beta(1)} : C_p(X_{\alpha(1)}) \rightarrow C_p(Y_{\beta(1)})$ densely defined on the linear span $D(T_{\alpha(1),\beta(1)})$ of D_1 such that

$$T(f_{\alpha(1)} \circ p_{\alpha(1)}) = T_{\alpha(1),\beta(1)} \circ q_{\beta(1)} \quad (2.11)$$

for each $f_{\alpha(1)} \in D(T_{\alpha(1),\beta(1)})$. By (2.10), (2.11), and the surjectivity of $q_{\beta(1)}$ we see that $T_{\alpha(1),\beta(1)}(f_{\alpha(1),i}) = g_i$ for each i . Thus we have the inclusion $R(T_{\alpha(1),\beta(1)}) \supset q_{\beta(1)}^\#(E)$.

Let us summarize the above properties of $T_{\alpha(1),\beta(1)}$:

$$(1.1) \quad D(T_{\alpha(1),\beta(1)}) \supset p_{\alpha(1)}^\#(D(T_{\alpha,\beta})) \text{ and}$$

$$(1.2) \quad R(T_{\alpha(1),\beta(1)}) \supset q_{\beta(1)}^\#(E).$$

We continue this process to obtain increasing sequences of indices $\alpha(1) < \dots < \alpha(n) < \dots$, $\beta(1) < \dots < \beta(n) < \dots$, countable dense subsets D_n of $C_p(X_{\alpha(n)})$, E_n of $C_p(Y_{\beta(n)})$, and a sequence of linear continuous operators $\{T_{\alpha(n),\beta(n)} : C_p(X_{\alpha(n)}) \rightarrow C_p(Y_{\beta(n)})\}$, each $T_{\alpha(n),\beta(n)}$ being defined on $D(T_{\alpha(n),\beta(n)})$, such that

$$(n.1) \quad D(T_{\alpha(n+1),\beta(n+1)}) \supset p_{\alpha(n)\alpha(n+1)}^\#(D(T_{\alpha(n),\beta(n)})),$$

$$(n.2) \quad R(T_{\alpha(n+1),\beta(n+1)}) \supset q_{\beta(n)\beta(n+1)}^\#(E_n), \text{ and}$$

$$(n.3) \quad T_{\alpha(n+1),\beta(n+1)} \circ p_{\alpha(n)\alpha(n+1)}^\# = q_{\beta(n)\beta(n+1)}^\# \circ T_{\alpha(n),\beta(n)} \text{ on } D(T_{\alpha(n),\beta(n)}).$$

Let $\alpha_\infty = \sup_n \alpha(n)$ and $\beta_\infty = \sup_n \beta(n)$ and let $D_\infty = \cup_n p_{\alpha(n)\alpha_\infty}^\#(D(T_{\alpha(n),\beta(n)}))$. Then

$$D_\infty \text{ is dense in } C_p(X_{\alpha_\infty}). \quad (2.12)$$

Indeed $X_{\alpha_\infty} = \lim_{\leftarrow} X_{\alpha(n)}$ by the ω -continuity (C3). Hence for each $f \in C_p(X_{\alpha_\infty})$ and for each $\epsilon > 0$, there exist $\alpha(n)$ and $f_{\alpha(n)} \in C_p(X_{\alpha(n)})$ such that

$$\sup_{x_\infty \in X_{\alpha_\infty}} |f(x_\infty) - f_{\alpha(n)}(p_{\alpha(n)}(x_\infty))| < \epsilon.$$

The function $f_{\alpha(n)}$ is approximated arbitrarily closely by functions of $D(T_{\alpha(n),\beta(n)})$ with respect to the pointwise convergence topology on $C_p(X_{\alpha(n)})$. Hence the function f is approximated arbitrarily closely by functions from the set $\cup_n p_{\alpha(n)\alpha_\infty}^\#(D(T_{\alpha(n),\beta(n)})) = D_\infty$.

An operator $T_{\alpha_\infty, \beta_\infty}$ is defined by

$$T_{\alpha_\infty, \beta_\infty}(p_{\alpha(n), \alpha_\infty}^\#(f_{\alpha(n)})) = T_{\alpha(n), \beta(n)}(f_{\alpha(n)}) \circ q_{\beta(n), \beta_\infty}$$

for $f_{\alpha(n)} \in D(T_{\alpha(n), \beta(n)})$. The equality (n.3) guarantees that $T_{\alpha_\infty, \beta_\infty}$ is well defined. Now we have the inclusion $T_{\alpha_\infty, \beta_\infty}(p_{\alpha_\infty \alpha(n)}^\#(D(T_{\alpha(n), \beta(n)}))) \supset q_{\beta(n), \beta_\infty}(E_n)$. Hence $R(T_{\alpha_\infty, \beta_\infty}) \supset E_\infty := \cup_n q_{\beta(n), \beta_\infty}^\#(E_n)$. Using the same argument as in (2.12) we see that E_∞ is dense in $C_p(Y_{\beta_\infty})$.

This proves (2).

Proof of Theorem 1.1. Let $T : C_p(X) \rightarrow C_p(Y)$ be a linear continuous surjection where X and Y are compact Hausdorff spaces. Let $\mathcal{S}_X = \{X_\alpha, p_{\alpha_1 \alpha_2}; A\}$ and $\mathcal{S}_Y = \{Y_\alpha, p_{\beta_1 \beta_2}; B\}$ be factorizing ω -spectra such that $X = \lim_{\leftarrow} \mathcal{S}_X$ and $Y = \lim_{\leftarrow} \mathcal{S}_Y$. By the assumption $\dim X = 0$, we may assume that $\dim X_\alpha = 0$ for each $\alpha \in A$. For each $\beta \in B$, we apply Proposition 2.2 to find a densely defined operator $T_{\alpha, \beta(\alpha)} : C_p(X_\alpha) \rightarrow C_p(Y_{\beta(\alpha)})$ such that $\beta(\alpha) \geq \beta$ and $R(T_{\alpha, \beta(\alpha)})$ is dense in $C_p(Y_{\beta(\alpha)})$. By Proposition 2.1, we have $\dim Y_{\beta(\alpha)} = 0$ and hence the set $\{\beta \in B \mid \dim Y_\beta = 0\}$ forms a cofinal subset of B . We therefore obtain $\dim Y = 0$. This completes the proof of Theorem 1.1.

Remark 2.4 *Theorem 1.1 holds under a weaker hypothesis that X is a zero-dimensional pseudocompact Tychonoff space and Y is a compact Hausdorff space.*

To see the above, assume that X is such a space and $T : C_p(X) \rightarrow C_p(Y)$ is a linear continuous surjection onto $C_p(Y)$ where Y is compact. The inclusion $h : X \rightarrow \beta X$ of X into the Stone-Ćech compactification βX of X induces a linear continuous surjection $h^\# : C_p(\beta X) \rightarrow C_p(X)$. Since $\dim \beta X = 0$, we may apply Theorem 1.1 to the composition $T \circ h^\#$ to conclude that $\dim Y = 0$.

Proof of Proposition 1.2. Let X be a Bing compactum, let Y be a hereditarily locally connected compact metrizable space, and let $T : C_p(X) \rightarrow C_p(Y)$ be a densely defined linear continuous operator with a dense image. First recall that each continuous map $\varphi : Z \rightarrow B$ of a locally connected continuum Z to a Bing compactum B must be a constant map. Proceeding as in the proof of Proposition 2.1, we see that the space Y is the countable union $Y = \cup_{i=1}^\infty Y_i$

such that each Y_i is homeomorphic to a subspace \tilde{Y}_i of $(\mathbb{R}^*)^n \times X^n$, and such that the projection $p_i : \tilde{Y}_i \rightarrow X^n$ is a finite-to-one map.

By the assumption, each component C of Y_i is locally connected. Applying the above remark to the composition of the map $p_i|_C : C \rightarrow X^n$ with the projection $X^n \rightarrow X$ onto any factor, we see that $p_i|_C$ must be a constant map. This implies that C is contained in a fiber of p_i , hence C is a finite set and thus C is a singleton. This proves that Y_i is totally disconnected which is equivalent to $\dim Y_i = 0$ by the compact metrizable of Y_i . By the countable sum theorem [5, Theorem 3.1.8], we conclude that $\dim Y = 0$.

3 Remarks and Problems

The proof of Proposition 2.1 relies on both the compactness and the metrizable of the spaces involved, while Remark 2.4 naturally raises the following problem.

Problem 3.1 *Does Theorem 1.1 hold for Tychonoff spaces X and Y ?*

Recently Krupski and Marciszewski [6] proved that there exists a metrizable space X such that $C_p(X)$ is not homeomorphic to $C_p(X) \times C_p(X) \simeq C_p(X \oplus X)$, negatively answering a long standing problem of Arkhangel'skii. In the paper above they also showed that there exists no linear continuous surjection $C_p(M) \rightarrow C_p(M) \times C_p(M)$ for the Cook continuum M . Recall that the Cook continuum M is a hereditarily indecomposable continuum such that, for each non-degenerate subcontinuum C of M , every continuous map $f : C \rightarrow M$ is either the identity map id or a constant map [4]. Since $M \oplus M$ is a subspace of $M \times M$, we see that there exists no linear continuous surjection $C_p(M) \rightarrow C_p(M \times M)$. On the other hand as has been pointed out by Marciszewski (a private communication), we have the following.

Remark 3.2 *For the pseudo-arc P , there exists a topological linear isomorphism $C_p(P) \simeq C_p(P \oplus P)$.*

Here we give a sketch of the above. Take a non-degenerate subcontinuum Q of P and let P/Q be the quotient space obtained from P by shrinking Q into a point. The space P/Q is a metrizable continuum as well. The projection $P \rightarrow P/Q$ is a monotone map, and monotone maps preserve the hereditary indecomposability and the arc-likeness (that is, being represented

by the limit of an inverse sequence of $[0,1]$, from the characterization of the pseudo-arc, we see that P/Q is homeomorphic to P . Then we obtain, by [15, Corollary 6.6.13], the following linear topological isomorphisms

$$C_p(P) \simeq C_p(P/Q) \times C_p(Q) \simeq C_p(P \oplus P).$$

This proves the desired result.

The following problem naturally arises.

Problem 3.3 *Does there exist a linear continuous surjection $C_p(P) \rightarrow C_p(P \times P)$?*

Here we notice that, if a finite-dimensional compact metrizable space X contains a homeomorphic copy of $[0,1]$, then there does exist a linear continuous surjection $C_p(X) \rightarrow C_p(X \times X)$. In order to see this, first observe that an arbitrary embedding $h : [0,1] \rightarrow X$ induces a linear continuous surjection $h^\# : C_p(X) \rightarrow C_p([0,1])$. Since X is finite-dimensional, compact and metrizable, there exists a linear continuous surjection $T : C_p([0,1]) \rightarrow C_p(X)$ [8]. Then the composition $T \circ h^\#$ is the desired linear continuous surjection.

Remark 3.4 *Theorem 1.1 does not hold when we drop the assumption of linearity of the operator T .*

In fact we can prove the following: Let X be a compact metrizable space and let S be the convergent sequence, that is, the space homeomorphic to $\{0\} \cup \{\frac{1}{n} \mid n \in \omega\}$. Then there exists a continuous surjection $H : C_p(S) \rightarrow C_p(X)$.

Proof. The argument below is extracted from [6, Proposition 5.4]. The space $C_p(S)$ contains a closed homeomorphic copy of J , the space of irrationals. Since the Banach space $C(X)$ of all real-valued continuous functions with sup norm is separable, we obtain a continuous surjection $J \rightarrow C(X)$ which extends to a continuous surjection $H : C_p(S) \rightarrow C(X)$. The map H , regarded as a map to $C_p(X)$, is the desired continuous surjection $H : C_p(S) \rightarrow C_p(X)$.

References

- [1] A.V. Arkhangel'skii, *Topological function spaces*, Mathematics and Its Appl. (78), Kluwer Acad. Pub.1992.
- [2] A.V. Arkhangel'skii, *Problems in C_p -theory*, in J.van Mill and G.M. Reed eds., Open Problems in Topology, North-Holland (1990), 601-615.
- [3] A. Chigogidze, *Inverse Spectra*, North-Holland Math. Lib. 53 (1996), Elsevier.
- [4] H. Cook, *Continua which admit only the identity mapping onto non-degenerate continua*, Fund. Math. 60 (1967), 241-249.
- [5] R. Engelking, *Theory of Dimensions, Finite and Infinite*, Sigma Ser.in Pure Math. 10, Helderman Verlag (1995).
- [6] M. Krupski and W. Marciszewski, *A metrizable X with $C_p(X)$ not homeomorphic to $C_p(X) \times C_p(X)$* , preprint.
- [7] A. Leiderman, M. Levin and V. Pestov, *On linear continuous open surjections of the spaces $C_p(X)$* , Top. Appl. 81 (1997), 269-279.
- [8] A. Leiderman, S. Morris and V. Pestov, *The free abelian topological group and the free locally convex space on the unit interval*, J. London Math. Soc.,56 (1997), 529-538.
- [9] M. Levin, *A property of $C_p[0, 1]$* , Trans. Amer. Math. Soc., 363 (2011), 2295-2304.
- [10] W. Lewis, *The pseudo-arc*, Bol. Soc. Mat. Mexicana 5 (1999), 25-77.
- [11] S.B. Nadler, *Hyperspaces of sets*, Monographs and Texts in Pure and Appl. Math. 49, Marcel-Dekker (1978).
- [12] D. Pavlovskii, *On spaces of continuous functions*, Soviet Math. Dokl. 22 (1980), 34-37.
- [13] V. Pestov, *The coincidence of the dimension \dim of ℓ - equivalent topological spaces*, Soviet Math. Dokl, 28 (1982), 380-383.

- [14] E.V. Shchepin, *Topology of limit spaces of uncountable inverse spectra*, Russian Math. Surveys 31 (1976), 155-191.
- [15] J. van Mill, *The infinite-dimensional topology of function spaces*, North Holland Math. Lib. 64 (2001), Elsevier.

Kazuhiro Kawamura
Institute of Mathematics
University of Tsukuba
Tsukuba, Ibaraki 305-8071
Japan
kawamura@math.tsukuba.ac.jp

Arkady Leiderman
Department of Mathematics
Ben-Gurion University of the Negev
Beer-Sheva
Israel
arkady@math.bgu.ac.il